Classification of Voronoi and Delone tiles in quasicrystals: I. General method

2003 J. Phys. A: Math. Gen. 36 1869

(http://iopscience.iop.org/0305-4470/36/7/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 1869–1894 PII: S0305-4470(03)39942-1

Classification of Voronoi and Delone tiles in quasicrystals: I. General method

Z Masáková[1], J Patera[2] and J Zich[3]

[1] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Trojanova 13, 120 00 Praha 2, Czech Republic
[2] Centre de Recherches Mathématiques, Université de Montréal, CP 6128, Succursale Centre-Ville, Montréal, Québec, Canada H3C 3J7
[3] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Trojanova 13, 120 00 Praha 2, Czech Republic

E-mail: masakova@km1.fjfi.cvut.cz, patera@crm.umontreal.ca and zich@kaktus.cz

Received 24 July 2002, in final form 2 December 2002
Published 5 February 2003
Online at stacks.iop.org/JPhysA/36/1869

Abstract

A new general method is presented which allows one to find all distinct Voronoi and Delone tiles in any quasicrystal from a large family. This includes the tiles which may be present with arbitrarily low density >0. At all stages, the method requires only consideration of a (possibly large) finite number of cases. Our method is applicable, in principle, to quasicrystals in any dimension and with any irrationality.

This is the first of three papers where the Voronoi and Delone tilings are studied. Two-dimensional point sets, 'quasicrystals', arising from the A_4-root lattice by means of the standard projection to a two-dimensional plane with the irrationality $\tau = \frac{1}{2}(1+\sqrt{5})$, are considered. In general, we require that the acceptance window be bounded with non-empty interior. Specific results are provided here for rhombic acceptance windows of any size oriented along the direction of simple roots of the Coxeter group H_2. Within one quasicrystal the tiles are distinguished by their shape, size and orientation. The rhombic window case is indispensable for subsequent classification of Voronoi and Delone tiles in quasicrystals with general shape of the acceptance window. Voronoi and Delone tiles of quasicrystals with circular and decagonal windows of any size are given in subsequent papers.

Let VT denote the set of distinct Voronoi tiles making up a quasicrystal with a given acceptance window. There are three VT sets of the 'generic' type and three of the 'singular' type. The latter occur for one precise value of the size of the acceptance window. Any other VT set is a uniform scaling of the tiles listed here. Similar results, differing in detail, are provided for the sets of distinct Delone tiles DT. Altogether there are four different sets DT of Delone tiles.

PACS number: 61.44.Br

1. Introduction

Uniform tiling of an Euclidean plane $\mathbb{R}^2$ by a small number of distinct tiles has a history stretching back centuries, indeed millennia. For a long time all such tilings were periodic. Only in recent decades have uncountable families of aperiodic tilings become known [6, 23]. In this paper we are concerned with a pair of tilings (Voronoi and Delone tiling), arising naturally from aperiodic point sets commonly used as idealized models of physical quasicrystalline materials; hence for simplicity we call such sets *quasicrystals* here.

The quasicrystals in this paper belong to a large family of infinite point sets which are deterministic, lack any translationally invariant subsets, and are uniformly dense and uniformly discrete. The sets arise through a well-known process called the cut-and-project method, starting from a higher dimensional lattice. The projection introduces a certain irrationality into the coordinates of the points and the cut is determined by a chosen bounded region called an acceptance window.

Voronoi domains and/or their duals, Delone domains, of a discrete point set in $\mathbb{R}^n$ are used in diverse and often unrelated fields as is witnessed by their different names: Dirichlet domains, proximity cells, Wigner–Seitz cells, Brillouin zones and so on. Each field has its specific applications and problems related to the properties of the particular set of points which defines the domains [21].

For the study of physical properties of quasicrystalline materials, e.g. electron conductivity or interatomic interactions, knowledge of local configurations of particles in the material is essential. In a mathematical model of quasicrystals, one thus needs to define neighbours of a point in the point set. In one dimension the notion of neighbours is clear. In two dimensions and higher, a natural definition of neighbours uses Voronoi domains of the modelling point set. Voronoi domains of all its points then form a perfect tiling of the space. For construction of the Delone domains meeting in the point X, one also needs to know about the second nearest neigbours of X or adjacent Voronoi tiles. In some problems one may need to know local configurations involving the third, fourth and higher neighbours or, equivalently, larger clusters of Voronoi and/or Delone tiles. Adaptation of our method to such problems is straightforward but requires more computing. In physics, the most interesting applications are found in two and three dimensions, $n = 2, 3$. In coding theory higher dimensions are also encountered. Among the papers that have focused on the description of local configurations in quasicrystal models are [1, 2, 4].

Our goal here is to describe and to exemplify a method of determining complete lists of distinct Voronoi and Delone tiles found in a cut-and-project point set with the acceptance window of a given shape, denoted here by *VT* and *DT* respectively. It is known that such a list is finite for all the quasicrystals we consider, and that corresponding tiles, Voronoi or Delone, are convex polygons. It turns out that a more general problem can be solved at the same time, namely to find *VT* and *DT* in the case that the acceptance window is of a given shape and of *any* size.

Classification of distinct Voronoi and Delone tiles in a given point set Σ is rarely an easy task in dimension $n > 1$, even if one considers only deterministic sets with long range order. The problem has some interesting aspects even in periodic cases where there is just one shape of Voronoi/Delone tiles when the dimension of the space exceeds 3. The description of faces of all dimensions becomes a challenging tasks. For example, the Voronoi tiles of root lattices of simple Lie groups were described only a decade ago [18, 19, 26, 27].

The present paper is the first of a series of three papers [13, 14]. Its goal is twofold: (i) to describe a new rigorous and rather general approach to the problem, and (ii) to provide a complete answer to the problem for a family of specific cases, namely two-dimensional cut-and-project point sets with the acceptance window of the shape of an equilateral parallelogram of any size. Interest in this case is more than just a curiosity. Its solution is indispensable for solving the classification problems in the case of an acceptance window of general shape. In subsequent papers, *VT* and *DT* are found where the acceptance window is a circular disk [13] and a regular pentagon [13] of any size.

The discovery of quasicrystals in 1982 [25] incited greater interest in aperiodic tilings with pentagonal (or icosahedral) symmetries [3, 8, 9, 22]. Some of the simplest aperiodic tilings, such as the Penrose pentagonal quasicrystals, are out of our consideration here: they are neither Voronoi nor Delone type, although they sometime can be cut further to become such. In a number of cases properties of related point sets, such as cut-and-project point sets, are not available in the literature.

There is an uncountable number of deterministic pentagonal quasicrystals, arising in the so-called cut-and-project method, for which the Voronoi and Delone domains have never been completely listed before, although properties of their local configurations have often been considered and partial lists frequently compiled from a finite size fragment(s) of a quasicrystal. Such compilations provide only crude information about the relative density of different tiles, and no information whatsoever about the change of the density as a function of the size of the acceptance window. In particular, no singular two-dimensional tiling, occurring just for one size of the window, has apparently been noted before.

Within the Voronoi or Delone tiling of a given quasicrystal, two tiles are considered distinct if either their shapes, or sizes, or orientations are different. Tiles of the same shape but different size are always present. Various orientations of the same tile can often be described more succinctly by the symmetry properties of the acceptance window.

One of the results of this paper, as well as of [13, 14], is the conclusion that, in the most common family of such two-dimensional quasicrystals, Voronoi tiles with very low density are not exceptional. In fact, it is easy to point out any number of quasicrystals containing Voronoi and/or Delone tiles with densities >0, but still lower than any value fixed in advance. Consequently, one cannot conclude that a list of distinct tiles is complete if it was obtained from an observation of a finite fragment of the quasicrystal point set. Hence a different rigorous approach is needed. Based on the properties of the cut-and-project method, one can find a procedure to determine all Voronoi domains which occur in a given quasicrystal. The problem requires a new approach as well as demanding computer verification of a crucial finite set of cases.

The strategy of the method consists in first considering the one-dimensional quasicrystals. Consequently, we can easily solve the case of a two-dimensional quasicrystal given as a Cartesian product of two one-dimensional quasicrystals, here for simplicity called 'quasilattice'. Finally, the general problem becomes a finite one by suitable embedding of the general quasicrystal into a quasilattice and embedding of another quasilattice into the general quasicrystal.

Specific quasicrystals (cut-and-project point sets), considered here, are a result of the standard projection of points of the root lattice of type A_4 onto a two-dimensional subspace oriented in such a way that the irrationality $\tau = \frac{1}{2}(1 + \sqrt{5})$ arises. The relevant subgroup of the Coxeter group of A_4 is H_2. Thus we say that we consider only the quasicrystals of type H_2. The acceptance window in this paper is either a rhomb oriented along two roots of H_2, or a general bounded convex window with non-empty interior. The quasicrystal with rhombic acceptance window is the quasilattice mentioned above. It turns out that there are precisely six sets of Voronoi tiles, VT_j, $j = 1, \ldots, 6$ and four sets of Delone tiles DT_k, $k = 1, \ldots, 4$ and their uniform scalings by τ^m ($m \in \mathbb{Z}$), which occur in quasicrystals with a rhombic window of any size. The number of Voronoi tiles that are of different shape and size (but not orientation!) and appear simultaneously in a single tiling, varies from 4 to 12. The analogous numbers for the Delone tilings are 4 and 8.

It turns out that the acceptance window is divided into regions corresponding to different types of tiles. The area of a region determines the relative density of the corresponding tile in the tiling. Depending on the particular setup of the boundaries (open $\times$ closed) of the acceptance window, there can appear tiles with zero density with the corresponding region in the acceptance window of dimension <2. In this paper, we are interested in describing the tiles with positive density and therefore we consider the boundary of the rhombic acceptance window semi-closed; in this case the tiles of zero density do not appear.

There is an interesting analogy between the singular and nonsingular quasicrystals in two dimensions in this paper, and respectively the well-known two- and three-tile quasicrystals in one dimension, assuming that they have a connected segment for their acceptance window. In order to explain this analogy for quasicrystals of any dimension, we assume that the acceptance window is of fixed shape, convex, connected and of dimension n, and that its size is specified by a real number d. Furthermore, we assume that we can always find the set VT and/or DT of distinct Voronoi or Delone tiles in the quasicrystal. Consider now the sets VT and/or DT as functions of d, for $0 < d \leqslant \infty$. The sets, which remain unchanged for a finite (open) interval of values of d, form nonsingular tilings. Two different nonsingular sets are separated by a singular one which corresponds to an isolated value of d only.

After the mathematical preliminaries in section 2, we recall in section 3 properties of one-dimensional quasicrystals. Most of them are generally known. New and crucial for our method is proposition 3.6. In section 4 one finds a technically indispensable result, which is of some independent interest. It is the classification of VT and DT sets of tiles for quasicrystals formed as the product of two one-dimensional quasicrystals, i.e. quasilattices. The results are summarized in figures 3 and 4.

In section 5 we describe the method of determining all tiles of the Voronoi, respectively Delone, tiling of a given quasicrystal with general acceptance window. The method requires solution of a finite but very large problem. Subsequently, we intend to apply the method to quasicrystals with circular and decagonal acceptance windows [13, 14].

2. Preliminaries

2.1. Cut-and-project maps

Quasicrystal models considered in this paper are constructed using the well-known cut-and-project scheme. Let L be a crystallographic lattice in $\mathbb{R}^{2n}$. Let V_1 and V_2 be two n-dimensional subspaces of $\mathbb{R}^{2n}$ and let π_1, π_2 be projections $\pi_1 : \mathbb{R}^2 \to V_1$ and $\pi_2 : \mathbb{R}^2 \to V_2$ which satisfy the following:

(1) π_1 restricted to L is an injection.

(2) $\pi_2(L)$ is dense in V_2.

The scheme is illustrated in the following picture:

$$\begin{tikzcd} & L \subset \mathbb{R}^{2n} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V_1 \arrow[rr, "*"] & & V_2 \end{tikzcd}$$

In this scheme $\pi_1(L)$ and $\pi_2(L)$ are additive Abelian groups. The bijection between them, $\pi_2 \circ \pi_1^{-1}$, was named the star map in [5] (because in the case $L = A_4$ it twists a regular pentagon of H_2 roots into a five-pointed star).

The cut-and-project scheme is rather general. In this paper, we focus on the special case of two-dimensional quasicrystals with tenfold symmetry. An algebraic formalism defining the cut-and-project scheme explicitly is found in [16]. In order to be specific, we provide the relation between points of the root lattice $L(A_4)$ of the Coxeter group A_4 and the root lattices $L(H_2) = \pi_1(L(A_4))$ and $L^*(H_2) = \pi_2(L(A_4))$, the two copies of the root lattice of H_2 in $\mathbb{R}^4$, denoted by V_1 and V_2 respectively in the picture above.

Let $\tau = \frac{1}{2}(1+\sqrt{5})$, $\tau' = \frac{1}{2}(1-\sqrt{5})$. We denote by $\mathbb{Z}[\tau]$ the set $\mathbb{Z}[\tau] = \mathbb{Z} + \mathbb{Z}\tau$. It is the ring of integers of the quadratic extension $\mathbb{Q}[\sqrt{5}]$ of rational numbers $\mathbb{Q}$. The ring is dense in $\mathbb{R}$. The Galois automorphism, denoted by $'$ on $\mathbb{Q}[\sqrt{5}]$, is defined by $x = a + b\tau \mapsto x' = a + b\tau'$ for $a, b \in \mathbb{Q}$. The Galois automorphism is an everywhere discontinuous map.

Let us fix in $\mathbb{R}^4$ bases, consisting of lattice vectors, and identify them by the corresponding Gram matrices:

$$L(A_4) : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad L(H_2) : \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix} \qquad L^*(H_2) : \begin{pmatrix} 2 & -\tau' \\ -\tau' & 2 \end{pmatrix}.$$

We denote the bases respectively by $\{\gamma_1, \ldots, \gamma_4\}$, $\{\alpha_1, \alpha_2\}$ and $\{\alpha_1^*, \alpha_2^*\}$. Thus the first two bases consist of simple roots of A_4 and H_2 respectively, while α_1^* and α_2^* are roots of H_2 but not simple ones. Determinants of the Gram matrices are frequently present in our calculation, so that it is useful to introduce special symbols for their square roots:

$$\Delta^* = \sqrt{4-\tau^2} = \sqrt{2+\tau'} \quad \text{and} \quad \Delta = \sqrt{4-(\tau')^2} = \sqrt{2+\tau}. \tag{1}$$

The determinant of the Gram matrix of A_4 is equal to 5.

Following [16], we can write a generic point $X \in L(A_4)$ as follows,

$$X = \sum_{k=1}^{4} c_k\gamma_k = \frac{c_1 + \tau c_3}{\Delta}\alpha_1 + \frac{\tau c_2 + c_4}{\Delta}\alpha_2 + \frac{c_1 + \tau' c_3}{\Delta^*}\alpha_1^* + \frac{\tau' c_2 + c_4}{\Delta^*}\alpha_2^*$$

where $c_1, \ldots, c_4 \in \mathbb{Z}$. More generally, we can take $c_1, \ldots, c_4 \in \mathbb{Q}$, replacing the lattices L by the corresponding $\mathbb{Z}[\tau]$-modules. Explicitly, the bases are related as follows:

$$\gamma_1 = \frac{1}{\Delta}\alpha_1 + \frac{1}{\Delta^*}\alpha_1^* \qquad \gamma_2 = \frac{\tau}{\Delta}\alpha_2 + \frac{\tau'}{\Delta^*}\alpha_2^* \qquad \gamma_3 = \frac{\tau}{\Delta}\alpha_1 + \frac{\tau'}{\Delta^*}\alpha_1^* \qquad \gamma_4 = \frac{1}{\Delta}\alpha_2 + \frac{1}{\Delta^*}\alpha_2^*$$

$$\alpha_1 = \frac{1}{\Delta}(\gamma_1 + \tau\gamma_3) \qquad \alpha_2 = \frac{1}{\Delta}(\tau\gamma_2 + \gamma_4) \qquad \alpha_1^* = \frac{1}{\Delta^*}(\gamma_1 + \tau'\gamma_3) \qquad \alpha_2^* = \frac{1}{\Delta^*}(\tau'\gamma_2 + \gamma_4).$$

2.2. *Quasicrystal definition*

Mathematical models of quasicrystals arising in such a cut-and-project scheme are explicitly defined using the algebraic formalism introduced in [16]. For our purposes it suffices to formulate the definition for quasicrystals in two dimensions, i.e. for the case of H_2.

An essential tool of the definition as well as of practical construction of the quasicrystals is the star map acting between the subspaces V_1 and V_2. More precisely, we have the 'star map' as follows:

$$\begin{aligned} * : M = L(H_2) &\longleftrightarrow M^* = L^*(H_2) \\ x = x_1\alpha_1 + x_2\alpha_2 &\longleftrightarrow x^* = x_1'\alpha_1^* + x_2'\alpha_2^* \qquad x_1, x_2 \in \mathbb{Z}[\tau]. \end{aligned}$$

Note that M and M^* are dense in $\mathbb{R}^2$ and the star map is everywhere discontinuous.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set with non-empty interior. The cut-and-project quasicrystal is the set*

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\} \tag{2}$$

where Ω is called the acceptance window.

2.3. Properties of quasicrystals

Let us recall some useful properties of $\Sigma(\Omega)$. The first one is the Delone property of $\Sigma(\Omega)$. It means that $\Sigma(\Omega)$ is (i) uniformly discrete, i.e. there exists $r > 0$ such that $|x - y| \geqslant r$ for every $x, y \in \Sigma(\Omega)$, $x \neq y$, and (ii) relatively dense, i.e. there exists $0 < R < \infty$ such that every ball of radius R contains at least one point of $\Sigma(\Omega)$. The minimal R satisfying (ii) is called the covering radius of $\Sigma(\Omega)$, and we denote it by R_c. Roughly speaking, the Delone property ensures that the points of $\Sigma(\Omega)$ are spread more or less uniformly in the entire plane.

Another interesting property of cut-and-project quasicrystals is the limited number of distinct local configurations of their points. There are finitely few configurations of any given size [15]. More precisely, given $\rho > 0$ the family of sets

$$(\Sigma(\Omega) - x) \cap B(0, \rho) \qquad \text{for all} \quad x \in \Sigma(\Omega)$$

is finite. Here $B(z, r)$ denotes the ball of radius r centred at z.

Under some additional requirements on Ω, the quasicrystal $\Sigma(\Omega)$ is repetitive; that is, every finite pattern (a subset of points) of its points occurs infinitely many times in $\Sigma(\Omega)$. A sufficient condition for that to happen is that the boundary of Ω has an empty intersection with the module $M^* = M$. Otherwise points of the boundary (or their absence in the quasicrystal) may cause unique occurrence of certain configurations in the quasicrystal. We will later illustrate this phenomenon in the simple example of one-dimensional quasicrystals.

All occurrences of a given pattern $P \subset \Sigma(\Omega)$ are found in the following way. Let x be a point of the pattern, $x \in P \subset \Sigma(\Omega)$. The connected region Ω_P,

$$\Omega_P = \{y^* \in \Omega \mid y^* - x^* + P^* \subset \Omega\}$$

describes all possible shifts of P^* inside Ω. Then one has $y - x + P \subset \Sigma(\Omega)$.

The density of occurrence of any fixed finite pattern P of $\Sigma(\Omega)$ is defined as the limit of the ratio of the number of points x such that $x + P \subset \Sigma(\Omega)$ in a ball of radius ρ, as ρ tends to infinity. It is a consequence of results in [17] that the density is proportional to the volume of the region Ω_P relative to the volume of the entire window Ω.

2.4. Voronoi and Delone cells

The Voronoi cell of a point $x \in \Sigma(\Omega) \subset \mathbb{R}^2$ consists of all the points of $\mathbb{R}^2$ that are closer to x than to any other point of $\Sigma(\Omega)$,

$$V(x) := \{z \in \mathbb{R}^2 \,|\, |z - x| \leqslant |z - y| \text{ for every } y \in \Sigma(\Omega)\}$$

Thus every Voronoi cell contains precisely one quasicrystal point. It follows from the properties of cut-and-project quasicrystals that the Voronoi cell of every point of $\Sigma(\Omega)$ is a closed convex polygon.

It has been proved (for example in [24]) that the Voronoi cell $V(x)$ is determined by points of $\Sigma(\Omega)$ within the distance of $2R_c$ from x, i.e.

$$V(x) = \{z \in \mathbb{R}^2 \,|\, |z - x| \leqslant |z - y| \text{ for every } y \in \Sigma(\Omega) \cap B(x, 2R_c)\} \tag{3}$$

where $B(x, r)$ denotes the ball of radius r centred at x. This, together with the fact that there is only a finite number of local configurations of a given size, implies that there are only finitely many distinct Voronoi cells, up to a shift.

Voronoi cells of points in $\Sigma(\Omega)$ form a perfect tiling of $\mathbb{R}^2$: they fit side by side without overlaps and without unfilled space. A boundary is shared by the adjacent cells. One can naturally define adjacent points in $\Sigma(\Omega)$ by saying that points x and y are adjacent precisely if their Voronoi cells share a common edge. Connecting all the adjacent pairs of points in $\Sigma(\Omega)$, one derives the dual or Delone tiling of $\mathbb{R}^2$. The vertices of one Delone tile are those points of $\Sigma(\Omega)$ whose Voronoi tiles share a common vertex. This vertex is the centre of the excircle of the Delone tile. Let us now mention some details about perfect tilings.

2.5. *Tessellations of an Euclidean space*

Voronoi and Delone tilings of a plane are examples of tessellations of an Euclidean space $\mathbb{R}^n$ by a collection T of closed convex tiles. It is useful to underline some of the general properties that tessellations are endowed with. In this paper we are interested only in the case $n = 2$.

Elements (tiles) $X, Y \in T$ are said to be neighbours, provided $X \neq Y$ and their intersection $X \cap Y$ is non-empty of dimension m $(0 \leqslant m \leqslant n-1)$. We denote the neighbour relation and its negation respectively by

$$X \overset{m}{\bowtie} Y \quad (0 \leqslant m \leqslant n-1) \qquad X \not\bowtie Y.$$

It is symmetric,

$$X \bowtie Y \quad \Longleftrightarrow \quad Y \bowtie X$$

for any m, and non-reflexive

$$X \not\bowtie X \quad (X \text{ is not its own neighbour}).$$

For any $X \in T$, the set $N_1(X)$ of its first neighbours is finite,

$$N_1(X) = \bigcup_{m=0}^{n-1} N_1^m(X) \qquad N_1^m(X) = \{Y \in T \mid Y \overset{m}{\bowtie} X\}.$$

Here n is the dimension of the Euclidean space. Consequently, higher order neighbours are also finite. They are naturally introduced recursively:

$$N_{k+1}^m(X) = \bigcup_{Y \in N_k^m(X)} N_1^m(Y).$$

The tessellation T is connected because one has

$$T = \bigcup_{k=1}^{\infty} N_k^{n-1}(X) \qquad \text{for any} \quad X \in T.$$

Distance $d(X, Y)$ between $X, Y \in T$ is introduced as follows:

$$d(X, Y) = k \qquad k \in \mathbb{N}_0 \qquad \text{for all} \quad X \in T \quad Y \in N_k^{n-1}(X) \quad N_0^m(X) := \{X\}.$$

Thus T becomes a metric space.

In the Voronoi tiling of a Delone set we may identify the tile with the corresponding centre, i.e. there is a one-to-one correspondence between points and their Voronoi cells. Thus we can define neighbours $N_k(x)$, where x is a point in the Delone set.

3. One-dimensional quasicrystals

For our task we need first to bring together some, generally known, facts about one-dimensional cut-and-project quasicrystals and to complete them by proposition 3.6.

The definition of a one-dimensional quasicrystal $\Sigma(I)$ is the simplest special case of (2). For the acceptance window we have a bounded interval $I \subset \mathbb{R}$. We define

$$\Sigma(I) := \{x \in \mathbb{Z}[\tau] \mid x' \in I\}. \tag{4}$$

The distances between adjacent points of $\Sigma(I)$ are called tiles. They are determined by differences between points of *I*. Hence $\Sigma(I)$ can be viewed either as a tiling of the real axis, or as a sequence of the points (endpoints of the tiles).

Many facts about one-dimensional quasicrystals are known. First of all, $\Sigma(I)$ is a Delone set and thus its elements are naturally ordered into an increasing sequence $(y_n)_{n\in\mathbb{Z}}$.

Proposition 3.1 ([11])**.** *Let I be a bounded interval. Then*

$$\tau^k\Sigma(I) = \Sigma(\tau'^k I) \qquad k \in \mathbb{Z}. \tag{5}$$

Hence a rescaling of I by an integer power of τ' implies the rescaling of the tiles of $\Sigma(I)$ by the same power of τ. Thus (5) allows one to restrict the consideration to one-dimensional quasicrystals whose acceptance interval *I* has its length $|I|$ within a finite range, say, $\tau^{-1} < |I| \leqslant 1$. Any other quasicrystal can be brought to this case by a suitable rescaling according to (5).

Subsequently, we assume that *I* is a semi-closed interval. This requirement is sufficient for the corresponding one-dimensional quasicrystal to be a repetitive set. Note that adding/removing a boundary point may add/remove only one point of the quasicrystal. Had we considered the acceptance interval *I* closed or open, while both boundary points are in $\mathbb{Z}[\tau]$, we would encounter configurations which occur only once in the entire quasicrystal (i.e. have zero density).

Proposition 3.2 ([11])**.** *Let* $I = [c, c+d)$ *be an interval of length* $\tau^{-1} < d \leqslant 1$. *Let* $\Sigma(I) = \{y_n \mid n \in \mathbb{Z}\}$, *where* $(y_n)_{n\in\mathbb{Z}}$ *increases. If* $d = 1$ *then the distances* $y_{n+1} - y_n$, $n \in \mathbb{Z}$, *take two values,* τ *and* τ^2. *If* $\tau^{-1} < d < 1$, *then the distances* $y_{n+1} - y_n$, $n \in \mathbb{Z}$, *take three values* τ, τ^2 *and* τ^3.

It is useful to introduce a stepping function f that allows one to determine the neighbour of a point $x \in \Sigma(I)$ according to the position of x' in *I*.

Proposition 3.3. *Let* $I = [c, c+d)$ *be an interval of length* $\tau^{-1} < d \leqslant 1$. *Let* $\Sigma(I) = \{y_n \mid n \in \mathbb{Z}\}$, *where* $(y_n)_{n\in\mathbb{Z}}$ *is an increasing sequence. Define* $f : I \to I$ *by*

$$f(x) := \begin{cases} x + \tau'^2 & \text{for} \quad x \in [c, c+d-\tau'^2) \\ x + \tau'^3 & \text{for} \quad x \in [c+d-\tau'^2, c-\tau') \\ x + \tau' & \text{for} \quad x \in [c-\tau', c+d). \end{cases} \tag{6}$$

Then $f(y_n') = y_{n+1}'$.

The acceptance window $I = [c, c+d)$ is thus divided by the discontinuity points $\alpha = c + d - \tau'^2$, $\beta = c - \tau'$ of f into a disjoint union of three subintervals. The position of y_n' with respect to three discontinuity points determines which of the three values the difference $y_{n+1} - y_n$ takes.

Using proposition 3.2, one can identify every one-dimensional quasicrystal with a symbolic sequence $(t_n)_{n\in\mathbb{Z}}$ in a three-letter alphabet, say $\{S, M, L\}$, standing for short (S), middle (M) and long (L). More precisely, if $(y_n)_{n\in\mathbb{N}}$ is the increasing sequence of the quasicrystal $\Sigma(I)$, $\tau^{-1} < |I| \leqslant 1$, then we put

$$t_n = \begin{cases} S & \text{if} \quad y_{n+1} - y_n = \tau \\ M & \text{if} \quad y_{n+1} - y_n = \tau^2 \\ L & \text{if} \quad y_{n+1} - y_n = \tau^3. \end{cases} \tag{7}$$

Infinite 'words', associated with quasicrystals $\Sigma(I)$, are aperiodic, but reveal a certain level of ordering. One of the restrictive properties is their complexity. The complexity of a symbolic sequence is a function $\mathcal{C} : \mathbb{N} \to \mathbb{N}$ that assigns to $n \in \mathbb{N}$ the number of different blocks $t_i t_{i+1} \dots t_{i+n-1}$ of length n occurring in the sequence. The complexity of sequences derived from one-dimensional quasicrystals is linear, i.e. far less than $\mathcal{C}(n) = 3^n$ as one would expect from a random sequence in the three-symbol alphabet $\{S, M, L\}$. More precisely, we recall the following statement.

Proposition 3.4 ([7]). *Let $\mathcal{C}_d$ be the complexity function of the infinite word derived from the one-dimensional quasicrystal $\Sigma(I)$ with $I = [c, c+d)$, and let f be the step function (6):*

- *If $d \notin \mathbb{Z}[\tau]$, then*

$$\mathcal{C}_d(n) = 2n + 1 \qquad \textit{for} \quad n \in \mathbb{N}.$$

- *If $d \in \mathbb{Z}[\tau]$, then there exists a unique $k \in \mathbb{N}_0$ such that $f^{(k)}(a) = b$ or $f^{(k+1)}(b) = a$, where a, b are the discontinuity points of f. Consequently,*

$$\mathcal{C}_d(n) = \begin{cases} 2n+1 & \textit{for} \quad n \leqslant k \\ n+k+1 & \textit{for} \quad n > k. \end{cases}$$

Similarly as the discontinuity points a, b of f determine the occurrence of letters of certain type (words of length 1), the discontinuity points of the nth iteration $f^{(n)}$ of f determine the occurrence of words of length n in the infinite word. These discontinuity points divide the interval I into $\mathcal{C}_d(n)$ subintervals, each one of them corresponding to one word of length n (n consecutive letters).

The density of a particular n-tuple $w = u_i u_{i+1} \dots u_{i+n-1}$ in the infinite word $(t_n)_{n\in\mathbb{N}}$ is given by

$$\varrho_w = \lim_{k\to\infty} \frac{|\{i \in \mathbb{Z} \cap (-k, k) \mid t_i t_{i+1} \dots t_{i+n-1} = w\}|}{2k-1} \tag{8}$$

if the limit exists. For the case of the infinite word $(t_n)_{n\in\mathbb{N}}$ of a one-dimensional quasicrystal $\Sigma(I)$ the density is always well defined and it is proportional to the length of the subinterval of I corresponding to the chosen n-tuple. Since we consider I to be a semi-closed interval, all n-tuples have non-zero density and therefore are repeated infinitely many times in the word $(t_n)_{n\in\mathbb{Z}}$.

The set of distinct words

$$\mathcal{L}_n(d) = \{t_i t_{i+1} \dots t_{i+n-1} \mid i \in \mathbb{Z}\} \tag{9}$$

which are present in the infinite word $(t_n)_{n\in\mathbb{Z}}$, associated with one-dimensional quasicrystal $\Sigma(I)$, $I = [c, c+d)$, does not depend on the position of the acceptance window I, but only on its length. The complexity function is by definition the number of elements in that set, $\mathcal{C}(n) = |\mathcal{L}_n(d)|$. For example, $\mathcal{L}_1(d)$ is equal to the alphabet $\{S, M, L\}$ so that we have $\mathcal{C}(1) = 3$.

Our final aim in this section is to find out how much we can change $d = |I|$ without changing the set $\mathcal{L}_n(d)$. An answer is given in proposition 3.6. First however, we prove the following lemma.

Lemma 3.5. *Let $\tau^{-1} < d \leqslant 1$ and $I = [c, c+d)$. Let $n \in \mathbb{N}$. Then there exist d_1 and d_2 such that $\tau^{-1} \leqslant d_1 < d < d_2 \leqslant 1$ and*

$$\mathcal{L}_n(d) \subset \mathcal{L}_n(\tilde{d}) \qquad \textit{for all} \quad d_1 < \tilde{d} < d_2.$$

Proof. Let $t_i t_{i+1} \dots t_{i+n-1} \in \mathcal{L}_n(d)$. This n-tuple corresponds to a sequence of $(n+1)$ points, say $y_j, y_{j+1}, \dots, y_{j+n}$ in $\Sigma(I) = \{y_k \mid k \in \mathbb{Z}\}$, where $(y_k)_{k\in\mathbb{Z}}$ is increasing. Since $y_j, y_{j+1}, \dots, y_{j+n}$ all belong to $\Sigma(I)$, we must have

$$\max\{y'_j, y'_{j+1}, \dots, y'_{j+n}\} - \min\{y'_j, y'_{j+1}, \dots, y'_{j+n}\} < d.$$

Strict inequality is a consequence of the fact that $I = [c, c+d)$ is not a closed interval. Similar inequalities hold for all other n-tuples in $\mathcal{L}_n(d)$. Altogether we thus have $\mathcal{C}_d(n)$ strict inequalities for d. Clearly, we can find $d_1 < d$ that will satisfy these inequalities also. This implies that all n-tuples in $\mathcal{L}_n(d)$ are contained in $\mathcal{L}_n(d_1)$ and also in $\mathcal{L}_n(\tilde{d})$ for all $d_1 < \tilde{d} < d$.

Let us now show that we can also enlarge the acceptance window without losing an n-tuple. Suppose that there is an n-tuple $t_i t_{i+1} \dots t_{i+n-1} \in \mathcal{L}_n(d)$ that is not in $\mathcal{L}_n(\tilde{d})$ for any $\tilde{d}$. It means that any enlarging of the acceptance window would destroy all occurrences of this n-tuple, i.e. a new point would be added between points of any $(n+1)$-tuple $y_j, y_{j+1}, \dots, y_{j+n}$ which correspond to the word $t_i t_{i+1} \dots t_{i+n-1}$. Since this is true for any enlarging, it follows that even adding the right endpoint to the acceptance interval $I = [c, c+d)$ must break all of the infinitely many occurrences of the n-tuple. This cannot be done by adding a single point and thus we arrive at a contradiction. Consequently, there must exist a $d_2 > d$ such that $\mathcal{L}_n(d) \subset \mathcal{L}_n(\tilde{d})$ for all $d < \tilde{d} < d_2$. □

Using the above lemma, we can prove the following statement.

Proposition 3.6. *Let $n \in \mathbb{N}$ be fixed. Denote by $\mathcal{C}_d$ the complexity function of the infinite word derived from the one-dimensional quasicrystal $\Sigma(I)$ with $I = [c, c+d)$. Define*

$$\mathcal{D}_n = \{d \mid \tau^{-1} < d \leqslant 1, \mathcal{C}_d(n) < 2n+1\}.$$

Then elements of $\mathcal{D}_n$ divide the interval $(\tau^{-1}, 1]$ into a finite disjoint union of subintervals, such that $\mathcal{L}_n(d)$ is constant on each of these intervals.

Proof. First we have to show that $\mathcal{D}_n$ is a finite set. In order to determine its elements, we use proposition 3.4. We start by recalling that the complexity of $\Sigma(I)$ is the same for all positions of I. Thus we can assume $I = [0, d)$. For given $n \in \mathbb{N}$, we have to solve the equations

$$f^{(k)}(a) = b \quad \text{or} \quad f^{(k+1)}(b) = a \qquad k \leqslant n \qquad \text{for} \quad a = d - \tau'^2 \quad b = -\tau'. \tag{10}$$

Note that in these equations $d \in \mathbb{Z}[\tau]$ is unknown and f is also unknown, since it depends on d. However, f is piecewise linear with slope 1 and thus, for a fixed $k \in \mathbb{N}$, there are only finitely many different prescriptions for $f^{(k)}$. The equation therefore can be solved and has only finitely many solutions $d \in \mathbb{Z}[\tau]$.

Next select a $d \in (\tau^{-1}, 1)$, such that $d \notin \mathcal{D}_n$. This means that $\mathcal{C}_d(n) = 2n+1$. From lemma 3.5 we know that there exists an open interval containing d, such that $\mathcal{L}_n(d)$ is a subset of $\mathcal{L}_n(\tilde{d})$ for all $\tilde{d}$ in the interval. Clearly, since $|\mathcal{L}_n(d)| = \mathcal{C}_d(n) = 2n+1$, the inclusion is in fact an equality $\mathcal{L}_n(d) = \mathcal{L}_n(\tilde{d})$. So let us take the maximal open interval with that property, (d_1, d_2). In other words, no $d_3 < d_1$ has the property that $\mathcal{L}_n(d) \not\subset \mathcal{L}_n(\tilde{d})$ for any $\tilde{d} \in (d_3, d)$. Similarly, no $d_4 > d_2$ has the property that $\mathcal{L}_n(d) \not\subset \mathcal{L}_n(\tilde{d})$ for any $\tilde{d} \in (d, d_4)$.

Let us first consider d_1. We want to show that $d_1 \in \mathcal{D}_n$. Assume that this is not the case. Then $\mathcal{C}_{d_1}(n) = 2n+1$. According to lemma 3.5 there exists a $d_5 < d_1 < d_6$ such that $\mathcal{L}_n(d_1) \subset \mathcal{L}_n(\tilde{d})$ for any $\tilde{d} \in (d_5, d_6)$. Since $\mathcal{C}_{d_1}(n) = 2n+1$, we have $\mathcal{L}_n(d_1) = \mathcal{L}_n(\tilde{d})$

for any $\tilde{d} \in (d_5, d_6)$. But (d_5, d_6) and (d_1, d) have a non-empty intersection. Necessarily $\mathcal{L}_n(d_1) = \mathcal{L}_n(d) = \mathcal{L}_n(\tilde{d})$ for all $\tilde{d} \in (d_6, d)$. However, this is a contradiction with the fact that (d_1, d_2) was the maximal interval.

Thus we have shown that $d_1 \in \mathcal{D}_n$. In a similar way we can also prove $d_2 \in \mathcal{D}_n$. This implies that $\mathcal{L}_n(d)$ remains constant on the subintervals of $(\tau^{-1}, 1)$ determined by $\mathcal{D}_n$. □

Proposition 3.6 is used extensively in the next section for computation of singular cases of one-dimensional quasicrystal. For this one-dimensional quasicrystals need to be divided according to different elements of $\mathcal{L}_4(d)$, the set of different words of length 4. It is done by solving equations (10), where d is the unknown length for $k = 0, 1$ and 3. For such small values of k, it is not difficult to compute the values of d manually. There are the following solutions:

$$\begin{array}{lll} f^{(0)}(a) = f^{(0)}(b) = a = b & \text{for} & d = 1 \\ f^{(2)}(a) = b & \text{for} & d = 3\tau - 4 \\ f^{(2)}(b) = a & \text{for} & d = 4 - 2\tau. \end{array} \tag{11}$$

Thus we have three singular cases $d = 1$, $d = 3\tau - 4$ and $d = 4 - 2\tau$, where the complexity of the words of length 4 is lower than $2n + 1 = 2 \times 4 + 1 = 9$.

4. Classification of Voronoi clusters of quasilattices

An indispensable step in our considerations is the study of the simplest cases of two-dimensional cut-and-project quasicrystals, namely those which are the Cartesian product of two one-dimensional quasicrystals. For simplicity of their identification, we call them *quasilattices* here.

Let $\{\alpha_1, \alpha_2\}$ be a basis in the space V_1 of the quasicrystal; its star map $\{\alpha_1^*, \alpha_2^*\}$ is the corresponding basis in V_2, the space of the acceptance window. From the definition of cut-and-project quasicrystals, it is easy to see that if the acceptance window Ω is given by $\Omega = I\alpha_1^* + I\alpha_2^*$ for a bounded semi-opened interval I, then the corresponding two-dimensional quasicrystal $\Sigma(\Omega)$ can be written as

$$\Sigma(\Omega) = \Sigma(I)\alpha_1 + \Sigma(I)\alpha_2. \tag{12}$$

To be specific, we take $\{\alpha_1, \alpha_2\}$ to be the simple roots of H_2, and $\{\alpha_1^*, \alpha_2^*\}$ to be their star maps [5]. Realizing both the space of the quasicrystal and the space of the acceptance window as complex planes, we can set

$$\alpha_1 = 1 \quad \alpha_2 = \mathrm{e}^{2\pi \mathrm{i}/5} \qquad \text{and} \qquad \alpha_1^* = 1 \quad \alpha_2^* = \mathrm{e}^{4\pi \mathrm{i}/5}.$$

4.1. Voronoi tilings

In order to determine the Voronoi cells, one needs to know, according to (3), the covering radius of the quasicrystal point set. Recall the following known results, see e.g. [28].

Proposition 4.1. *Let I be a semi-opened bounded interval of length $d \in (\tau^k, \tau^{k+1}]$. The covering radius of the quasilattice $\Sigma(\Omega)$ for $\Omega = I\alpha_1^* + I\alpha_2^*$ is equal to*

$$R_c = \frac{\tau^{2-k}}{\sqrt{\tau + 2}}.$$

For practical construction of the Voronoi tiling it is advantageous to use another result: for determination of the Voronoi cell of a chosen point of $\Sigma(\Omega)$, it suffices to consider only its first and second neighbours in the quasilattice. More precisely, we have the following proposition.

Proposition 4.2. *Let I be a semi-opened bounded interval and let $\Sigma(I) = \{y_n \mid n \in \mathbb{N}\}$, where $(y_n)_{n\in\mathbb{N}}$ increases. Let $\Omega = I\alpha_1^* + I\alpha_2^*$. For the Voronoi domain $V(x)$ of the point $x = y_k\alpha_1 + y_\ell\alpha_2 \in \Sigma(\Omega)$ it holds that*

$$V(x) = \{z \in \mathbb{R}^2 \mid |z - x| \leqslant |z - y| \text{ for every } y \in \Sigma(\Omega) \cap S\}$$

where

$$S = \{y_i\alpha_1 + y_j\alpha_2 \mid k-2 \leqslant i \leqslant k+2, \ell-2 \leqslant j \leqslant \ell+2\}.$$

Proof. The proof of the above statement is constructed by inspection of a finite number of cases according to the following scheme.

First recall that, due to the rescaling property (5), it suffices to consider the acceptance window sides of length $|I|$ between two consecutive integer powers of τ. We choose $\tau^{-1} < |I| \leqslant 1$. For this case we have $R_c = \tau^3/\sqrt{\tau+2}$.

Next we want to estimate the size of rhombic fragment of the quasicrystal, oriented along α_1 and α_2, which would cover any disc of radius $2R_c$ centred on any point of the quasicrystal. Using simple geometrical formulae we get that the length of side of this rhombus is $8\tau R_c/\sqrt{\tau+2}$. It implies that we need one-dimensional sections of length at least $8\tau^4/(\tau+2)$. More precisely, we need to find the smallest even n such that

$$\begin{aligned}\forall d \in (1/\tau, 1] \quad \forall t_1 \dots t_n \in \mathcal{L}_{n+2}(d) \quad &\Longrightarrow \quad |t_1 \dots t_{n/2+1}| > 4\tau^4/(\tau+2)\\ &|t_{n/2+2} \dots t_{n+2}| > 4\tau^4/(\tau+2).\end{aligned}$$

Taking the shortest tiles S, we obtain an upper estimate for n. However, we know that sequence SS does not occur in quasicrystals, therefore a better estimate is obtained from $SMSM$. This gives $n = 3$. One has approximately $|SMS| \doteq 5.8541$ and the approximate value of $4\tau^4/(\tau+2)$ is 7.577 71. If we add one more M to the end we get the length approximately 8.472 14. Hence 6-tile words should be investigated. (Subsequent case-by-case investigation has shown that only 4-tile words determine the shape of the Voronoi tiles.)

We proceed by generating all possible sequences of the length 6 spaces which may appear in any one-dimensional quasicrystal $\Sigma(I)$ for which $1/\tau < |I| \leqslant 1$. This is done by applying the basic rules which forbid sequences that cannot occur in any quasicrystal, namely

$$\begin{gathered} SL, LS, SS, LLL, MML, LMM, MLMLML, MLMLMM,\\ LMLMLM, MMLMLM, MSMSMSMS, SMSMSMSM. \end{gathered} \tag{13}$$

Non-forbidden configurations were inspected by computer and in all cases only two neighbours on each side were necessary.

One may think that only the first neighbours suffice to determine the shape of a Voronoi tile. In many cases that is true. However, it is not difficult to find, by direct inspection of tilings, examples of Voronoi tiles determined by the first and the second neighbours. □

The Voronoi tiles of two quasilattices are the same, if distinct 4-tuples in the corresponding one-dimensional quasicrystals coincide. More precisely, let I_1 and I_2 be bounded intervals such that $\mathcal{L}_4(|I_1|) = \mathcal{L}_4(|I_2|)$. Let $\Omega_1 = I_1\alpha_1^* + I_1\alpha_2^*$ and $\Omega_2 = I_2\alpha_1^* + I_2\alpha_2^*$. Then the set of different Voronoi cells of $\Sigma(\Omega_1)$ and $\Sigma(\Omega_2)$ are the same. The quasilattices $\Sigma(\Omega_1)$ and $\Sigma(\Omega_2)$ differ only by the relative density of occurrence of Voronoi cells and by their tiling arrangements.

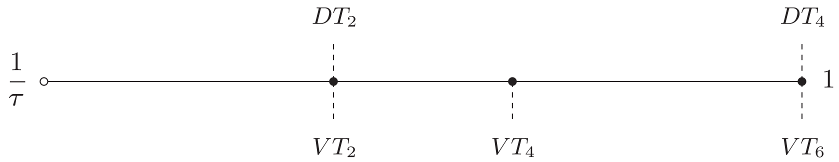


Figure 1. The positions of singular cases VT_2, VT_4 and VT_6 and DT_2, DT_4 within the range $(1/\tau, 1]$ of side lengths of the rhombic acceptance windows are shown. The figure is drawn to scale. Between two Voronoi/Delone singular cases the sets of Voronoi/Delone tiles do not change, only the relative density and arrangement of tiles in the tiling vary.

Table 1. The cases of quasicrystals with rhombic acceptance windows—quasi–lattices—specified by the sets of Voronoi and Delone tiles are given. There are six classes of quasicrystals $VT_m, m = 1, \ldots, 6$, which have different Voronoi tiles. The second and fourth columns contain the number of Voronoi and Delone tiles in the corresponding tiling. Sets VT_2, VT_4 and VT_6 are singular and they tile the quasicrystals with a specific size of window side; its length is shown in the middle column. There are only four sets of Delone tiles $DT_m, m = 1, \ldots, 4$ for quasicrystals with a rhombic acceptance window.

VT_1	12		8	DT_1
VT_2	8	$2/\tau^2$	6	DT_2
VT_3	12			
VT_4	10	$(\tau+2)/\tau^3$	6	DT_3
VT_5	12			
VT_6	4	1	4	DT_4

Different sets $\mathcal{L}_4(d)$ of 4-tuples in one-dimensional quasicrystals, as functions of the length d of the acceptance interval, are found using proposition 3.6, as was explained at the end of the previous section. Thus we find that one-dimensional quasicrystals split into six mutually exclusive types, according to the sets of local configurations (equivalently Voronoi tiles) they contain. Consequently, we can identify six types of two-dimensional quasilattices, according to the Voronoi tiles they generate. Three of them are singular in that they occur for a precise value of $|I|$, the length of the window-interval, three others are nonsingular, having $|I|$ within a finite range. More precisely, we obtain

$$\begin{array}{llll} VT_1: & |I| \in (\tau^{-1}, 4-2\tau) & VT_2: & |I| = 4-2\tau \\ VT_3: & |I| \in (4-2\tau, 3\tau-4) & VT_4: & |I| = 3\tau-4 \\ VT_5: & |I| \in (3\tau-4, 1) & VT_6: & |I| = 1. \end{array} \tag{14}$$

Results of (14) are shown in table 1 and figure 1 together with analogous information about the dual (Delone) tilings.

4.2. Delone tilings

So far we have been concerned about tiling of the Euclidean plane $\mathbb{R}^2$ by Voronoi tiles generated by a given quasilattice or, more generally, by a quasicrystal. Related to this is the problem of finding the Delone tiling for the same quasilattice/quasicrystal which we consider now. Our aim is to describe an algorithm that allows one to determine all Delone tiles.

Delone tiles are found in the following way. The process is illustrated in figure 2. Consider the midpoint c of a configuration of 5×5 points of a given quasicrystal $\Sigma(\Omega)$ together with the Voronoi tile $V(c)$. Find the finite set $N_1 = N_1^1 \cup N_1^0$ of tiles $V(x)$ for $x \in \Sigma(\Omega)$ with non-empty intersection $V(c) \cap V(x) \neq \emptyset$. The tiles of N_1 are said to be the

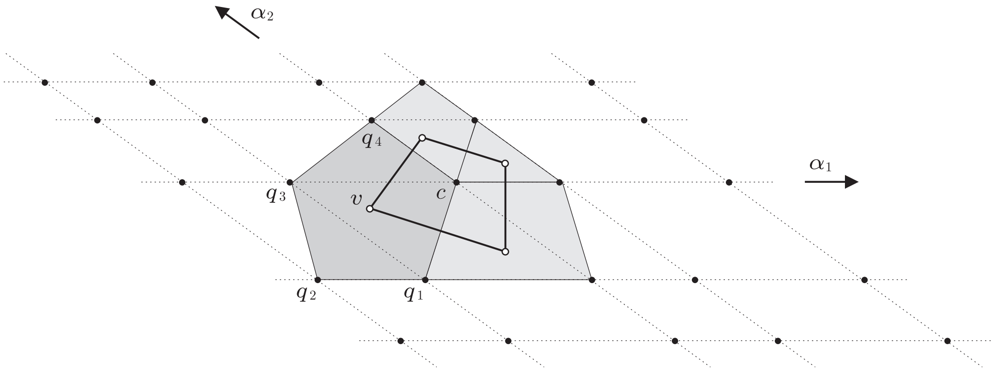


Figure 2. The Voronoi tile $V(c)$ and the cluster of four Delone tiles which have the quasicrystal point c in common.

first neighbours of $V(c)$. There are two types N_1^1 and N_1^0 of first neighbours, according to the dimension of the intersection $V(c) \cap V(x)$. We call them direct or degenerate neighbours, respectively.

The points q_1, q_2, q_3 and q_4 in figure 2 are the neighbours of c, corresponding to Voronoi vertex v. It means that the Voronoi tiles $V(q_i)$ share the vertex v with $V(c)$, or in other words, points $c, q_1, \ldots, q_4$ have the same distance from v. The neighbours q_2 and q_3 are degenerate. By definition, the Delone tile of the vertex $v \in V(c)$ is formed as the convex hull of exactly these points, q_1, q_2, q_4 and q_4 of figure 2.

Taking vertices of every Voronoi tile one-by-one, finding their direct and degenerate neighbours, and connecting those to Delone tiles, one builds the Delone tiling. From the configuration of 5×5 quasilattice points, containing c as its midpoint, we get the Voronoi tile $V(c)$ and the cluster of Delone tiles with c in common.

Between two singular cases VT_{2i} and VT_{2i+2}, the set of local configurations of the size 5×5 are the same. This also implies that the set of Delone tiles does not change. Hence, singular sizes of acceptance window for Delone tilings form a subset of singular sizes for Voronoi tilings.

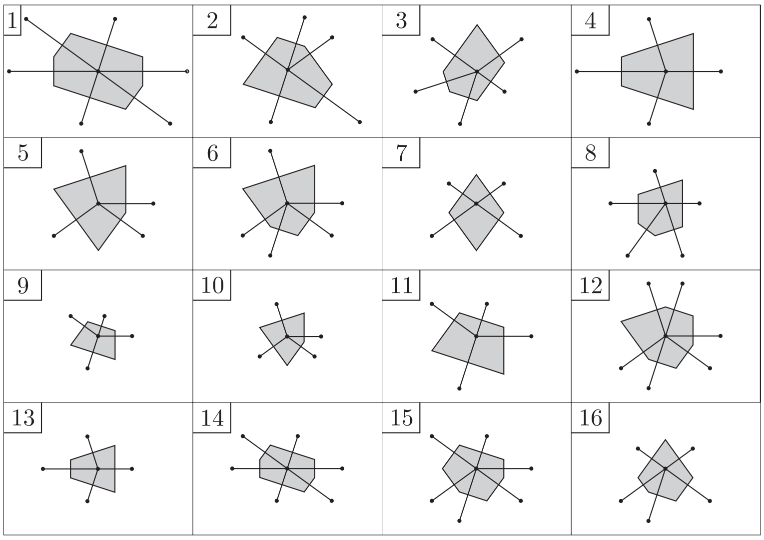
There are four sets DT_s, $s = 1, 2, 3, 4$ of distinct Delone tiles for the quasilattices (12). The correspondence between VT_m tiling sets and the DT_s is as follows:

$$\begin{array}{lll} DT_1 \longleftrightarrow VT_1 & & |I| \in (\tau^{-1}, 4 - 2\tau) \\ DT_2 \longleftrightarrow VT_2 & & |I| = 4 - 2\tau \\ DT_3 \longleftrightarrow VT_3 \cup VT_4 \cup VT_5 & & |I| \in (4 - 2\tau, 1) \\ DT_4 \longleftrightarrow VT_6 & & |I| = 1. \end{array} \tag{15}$$

In particular, Delone tilings corresponding to Voronoi cases VT_3, VT_4 and VT_4 are made out of the same set of tiles DT_3. The correspondence between the six cases of sets for Voronoi tiles and the four cases for Delone tiles is shown in detail in table 1 and graphically illustrated in figure 1.

4.3. The lists of tiles in VT_m and DT_m

Finally, let us describe the sets VT_m and DT_m of distinct Voronoi and Delone tiles in quasilattices (12) for all values of the length $|I|$ of the side of the acceptance window. The results are summarized in figures 3 and 4. Let us mention that the actual position of the window does not affect the set of Voronoi or Delone tiles that occur in (12).



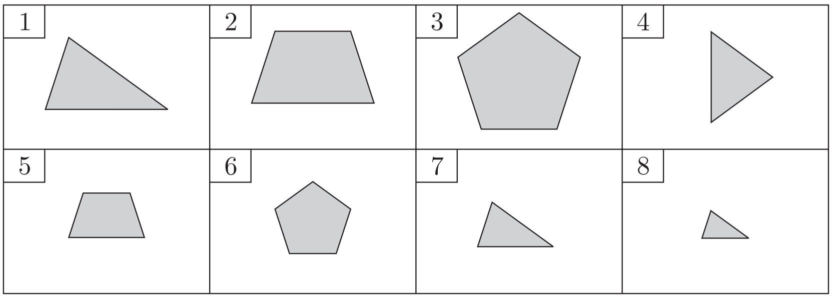
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
VT_1	•	•	•	•	•	•	•	•	•	•						
VT_2					•	•	•	•	•	•	•	•				
VT_3					•	•	•	•*	•	•	•	•	•	•	•	
VT_4					•	•	•	•*	•	•	•	•	•	•		
VT_5							•	•*	•	•	•	•	•	•	•	•
VT_6									•	•			•	•		

Figure 3. The top part contains a numbered list of all Voronoi tiles encountered in all quasicrystals with rhombic window—quasilattices. The tiles are shown together with the local configuration of quasicrystal points which shape them. Shapes and relative sizes of the tiles are maintained. The lower part indicates the subsets of tiles present in each tiling set $VT_1, \ldots, VT_m$. Each tile marked by a black dot occurs in four orientations given by the symmetries of the acceptance window. Tiles 1 and 14 have two axes of symmetry collinear with the quasicrystal symmetry axes and hence their four orientations coincide. Tiles 5, 9, 10, 11 have one symmetry axis collinear with the quasicrystal symmetry and hence occur in two different orientations only. Tile 8 is exceptional in that it occurs in eight different orientations in the cases denoted by an asterisk.

Figure 3 shows the shapes and relative sizes of all Voronoi tiles that may possibly occur in a quasilattice. Within a given Voronoi tile, we can see the position of the lattice point, and the picture also shows all the direct neighbours of the point.

In row 1 of the table in figure 3 individual tiles are numbered. Other rows of the table indicate the occurrences of the tiles in each case of VT_m sets. H_2-symmetry of the $\mathbb{Z}[\tau]$-module M together with the symmetries of the rhombic acceptance window ensure that tiles occur together with their symmetric copies. We now describe those symmetries.

The acceptance window is a parallelogram symmetric with respect to reflections in its diagonals; the $\mathbb{Z}[\tau]$-module M has the same symmetries. Denote the reflection in the longer diagonal by R_1 and the reflection along the shorter diagonal as R_2. By the star map one can verify that the symmetry transformations of m and the acceptance window ensure the symmetry of the quasicrystal with respect to the reflections along the bisections of both angles between α_1 and α_2. It is illustrated in figure 5.



	1	2	3	4	5	6	7	8
DT_1	•	•	•	•	•	•	•**	•
DT_2			•	•	•	•	•**	•
DT_3			•	•*	•	•	•**	•
DT_4					•	•	•	•

Figure 4. The table provides information analogous to that of the table in figure 3 for Delone tilings. There are exceptional tiles 4 and 7. The first case, denoted by *, consists in tile 4 occuring in tiling in eight orientations for the case DT_3. The second case, **, consists in tile number 7 occuring in tiling in six orientations for cases DT_1, DT_2 and DT_3.

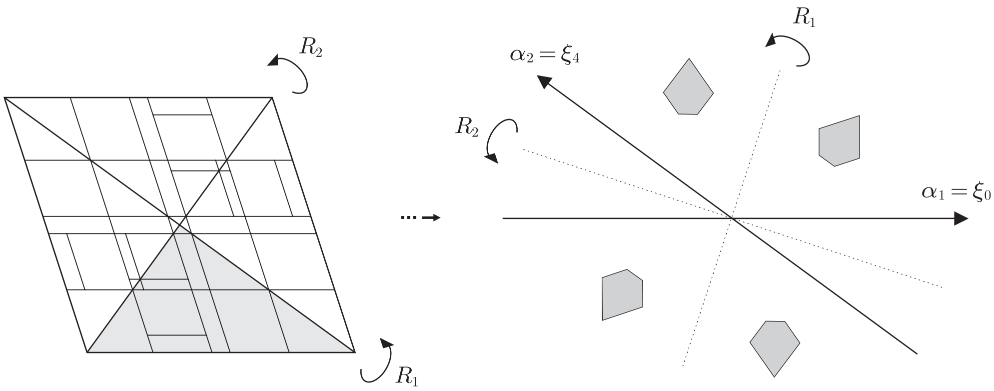


Figure 5. Reflection symmetry axes of a rhombic window of the quasilattice (12). An example of division of the window into regions where all points x^* with Voronoi tiles $V(x)$ of the same size and shape are found (on the left). Transformations of a Voronoi tile under the symmetries of the acceptance window (on the right).

The symmetry group $G(\Omega)$ of the window is of order 4. Through the star map, it also acts on the quasicrystal. Therefore a tile V occurs at most in four orientations, V, R_1V, R_2V and R_1R_2V, given by the action of elements of the group. If the symmetry group of a tile is a subgroup $G(V)$ of $G(\Omega)$, the number of distinct orientations of V is given by $|G(\Omega)|/|G(V)|$. Thus tiles 5, 9, 10 and 11, which are invariant under exactly one of the reflections R_1 or R_2, occur in two orientations in a tiling. Tiles 1 and 14 are invariant under both reflections and hence they occur only in one orientation. Tile number 8 has a symmetry group of order 2. Nevertheless the tile occurs in eight orientations in lists VT_3, VT_4 and VT_5 because its symmetry group is not a subgroup of $G(\Omega)$: its reflections have different orientations from those in $G(\Omega)$. More precisely, there are two isomorphic, but differently oriented, copies of tile 8, neither of which has orientations corresponding to R_1 or R_2. Each of the two copies gives rise to four tiles of shape 8.

4.4. The density of tiles in a quasilattice

In addition to the set of Voronoi tiles, we can gather more information about the position of individual Voronoi polygons in the acceptance window.

Consider a quasilattice $\Sigma(I \times I)$. With each Voronoi tile V of $\Sigma(I \times I)$ one can identify a region of the acceptance window. The area of that region is proportional to the density of the tile in $\Sigma(I \times I)$.

We have seen in section 3 that I can be written as the union of subintervals $I_1, I_2, \ldots, I_m$, each containing the star map of points of $\Sigma(I)$ with identical first and second neighbours.

Let $i, j \in \{1, \ldots, m\}$ and choose a point $x \in \Sigma(\Omega)$ such that x^* is in the rhombus with sides I_i and I_j,

$$x^\star \in I_i\alpha_1^* + I_j\alpha_2^\star.$$

Any other point $y \in \Sigma(\Omega)$ such that $y^* \in I_i\alpha_1^* + I_j\alpha_2^\star$ has the same Voronoi tile. It follows from the fact that x and y are surrounded by the same configuration of 5×5 points in the quasicrystal $\Sigma(\Omega)$. Their coordinates on axes α_1 and α_2 belong respectively to I_i and I_j. Thus the acceptance window—rhombus $\Omega = I \times I$—is divided into rhombuses that correspond to the different Voronoi polygons, star map images of Voronoi tiles.

These divisions of acceptance window are drawn in figures 6 and 7. In each region there is a number which denotes a corresponding tile from the table in figure 3. Note that symmetry of a tile under reflection R_1 or R_1 is equivalent to the fact that the corresponding region in the acceptance window lies on a diagonal.

Classification of quasilattices according to types of Delone tiles is also given in the table of figure 4 using the same conventions.

Figures 8 and 9 show Voronoi and Delone tiling of two cases of quasilattices. They illustrate a singular case with a low number of tiles ($|I| = 1$), and a non-singular case with a high number of tiles.

5. Voronoi clusters for quasicrystals with general acceptance window

In this section we want to describe how the sets VT and DT can be found for quasicrystals with an acceptance window Ω of general shape, assuming that it has a non-empty interior and a one-dimensional integrable boundary.

There are three ingredients of the present procedure where our results about the quasilattices are indispensable: (i) we know the covering radii R_c for quasilattices; (ii) points and local configurations of a quasilattice can be generated easily (unlike for a general quasicrystal); (iii) search for the required local configurations in a general quasicrystal uses the configurations in quasilattices.

First we take the smallest and the largest rhombic windows $\Omega_1 = I_1\alpha_1^\star + I_1\alpha_2^\star$ and $\Omega_2 = I_2\alpha_1^\star + I_2\alpha_2^\star$ with the property

$$\Omega_2 \subseteq \Omega \subseteq \Omega_1$$

which implies

$$\Sigma(\Omega_2) \subseteq \Sigma(\Omega) \subseteq \Sigma(\Omega_1).$$

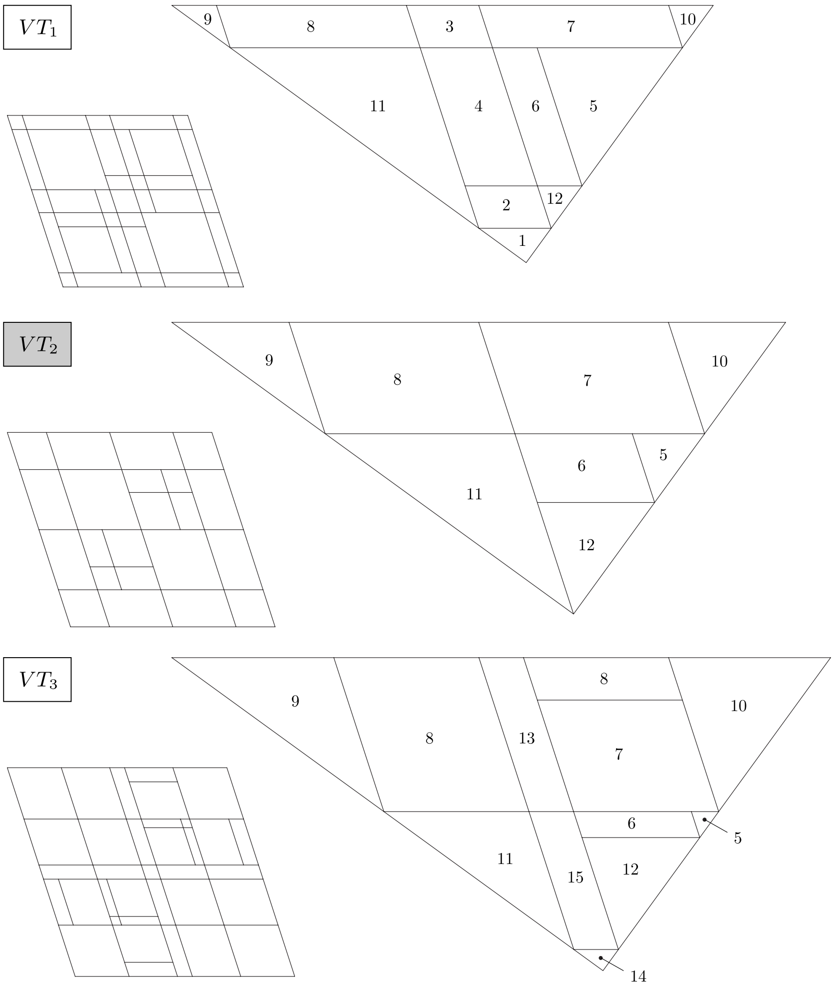


Figure 6. Division of rhombic acceptance windows for cases VT_1, VT_2 and VT_3. The numbering of the regions corresponds to the numbering of Voronoi tiles in the table of figure 3. Due to the symmetries of the acceptance window it suffices to show a triangular segment. Non-singular cases VT_1 and VT_3 are represented by the side lengths $d = 12 - 7\tau$ and $d = 17 - 10\tau$.

Thus the points of the quasicrystal $\Sigma(\Omega)$ form a subset of the quasilattice $\Sigma(\Omega_1)$. Since $\Sigma(\Omega_1)$ generally contains more points than $\Sigma(\Omega)$, we call $\Sigma(\Omega_1)$ a *dense quasilattice* and its points will be marked by empty circles, $\circ$. All points from $\Sigma(\Omega_2)$ are contained in $\Sigma(\Omega)$ and also in $\Sigma(\Omega_1)$. Therefore we call $\Sigma(\Omega_2)$ the *sparse quasilattice*. Points of $\Sigma(\Omega_2)$ will be marked by black dots, $\bullet$.

The quasicrystal $\Sigma(\Omega)$ has finite covering radius R_c because it is a Delone set. There are finitely many distinct local configurations of size $2R_c$ in $\Sigma(\Omega_1)$. Therefore there could be only finitely many configurations of that size in $\Sigma(\Omega)$. This fact implies that there are only finitely many Voronoi tiles in the tiling of $\Sigma(\Omega)$.

Our starting setup is as follows. We have embedded the quasicrystal $\Sigma(\Omega)$ into the quasilattice $\Sigma(\Omega_1)$, whose points are marked by either $\circ$ or $\bullet$. We know that the points marked by $\bullet$ are contained in $\Sigma(\Omega)$. In addition $\Sigma(\Omega)$ contains some other points $\circ$ from the dense

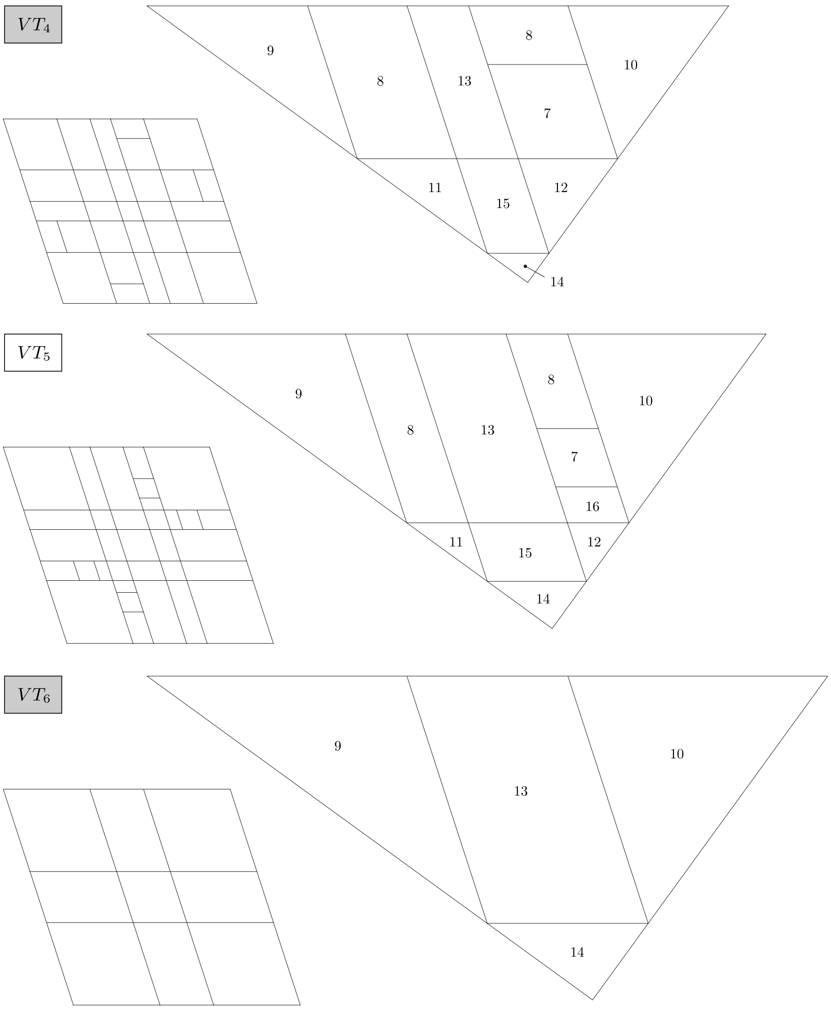


Figure 7. Division of rhombic acceptance windows for cases VT_4, VT_5 and VT_6. The non-singular case VT_5 is represented by the side length $d = 9 - 5\tau$.

quasilattice $\Sigma(\Omega_1)$. Subsequently in this paper, the point set $\Sigma(\Omega_1)$, decorated as $\circ$ and $\bullet$, is called a *skeleton*. An example of a skeleton quasilattice and the configuration of Ω_1, Ω_2 and Ω are illustrated in figure 10.

Our main problem is to find all Voronoi and then Delone tiles from the tilings of $\Sigma(\Omega)$. As in the previous section, one needs to know the covering radius of $\Sigma(\Omega)$ or at least an upper bound for it. Since all points of the quasilattice $\Sigma(\Omega_2)$ are elements of $\Sigma(\Omega)$, the covering radius of $\Sigma(\Omega)$ is smaller or equal to the covering radius of $\Sigma(\Omega_2)$. The covering radius of a quasilattice is given by proposition 4.1. We denote this upper bound by R_c.

In order to find all Voronoi tiles in the tiling of $\Sigma(\Omega)$, we have to determine all different configurations of points of $\Sigma(\Omega)$ which fit into the ball of radius $2R_c$. The first step is identification of local configurations of size $2R_c$ in the skeleton.

Intervals $I_2 \subseteq I_1$ define the skeleton in one dimension. The two-dimensional skeleton, given by Ω_1 and Ω_2, is the Cartesian product of one-dimensional ones. Therefore the clue

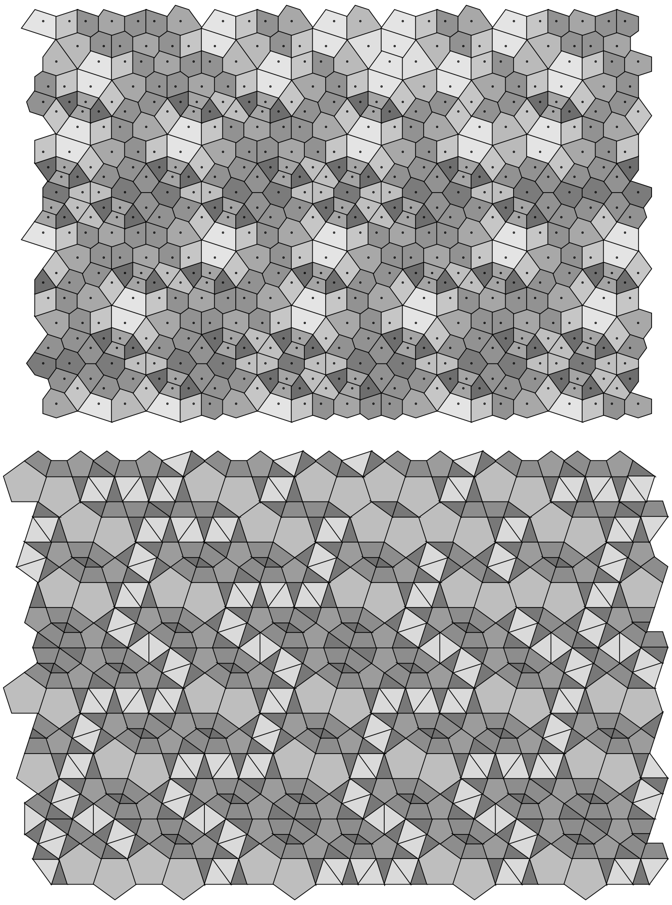


Figure 8. Example of a nonsingular Voronoi and Delone tiling: a quasilattice whose acceptance window has the side length $d = 17 - 10\tau$. It represents the cases VT_3 and DT_3.

is to find all finite sections from the one-dimensional skeleton given by I_1 and I_2 of length n, where n is to be determined. We denote by $\mathcal{L}_n(I_1, I_2)$ the set of all different words from skeleton given by I_1 and I_2 of the length n measured in spaces of $\Sigma(I_1)$. An algorithm for that is given in [28]. We proceed with the following steps:

(1) Embed a disc of the radius $2R_c$ into rhombus orientated along the axes α_1 and α_2. Let l be the length of the side of this rhombus.

(2) Find $n \in \mathbb{N}$ so that the length of the sequence of quasicrystals points, corresponding to every word from $\mathcal{L}_n(|I_1|)$, is at least $d/2$. This could be done by estimating the worst case. More precisely, it is sufficient to consider the word consisting of only the smallest possible spaces, avoiding the forbidden strings as named in formula (13).

(3) Compute the set of all different words $\mathcal{L}_{2n}(I_1, I_2)$.

(4) Every word from $\mathcal{L}_{2n}(I_1, I_2)$ is composed of $2n$ spaces. Now assign to the central point the coordinate 0 and compute the coordinates of the remaining points according to the spaces between them. Save the list of sequences as $\mathcal{Q}$.

(5) Compose the set $\tilde{\mathcal{N}}$ of configurations in $\mathbb{R}^2$ from the skeleton:

$$\tilde{\mathcal{N}} = \{q_1\alpha_1 + q_2\alpha_2 \mid q_1, q_2 \in \mathcal{Q}\}.$$

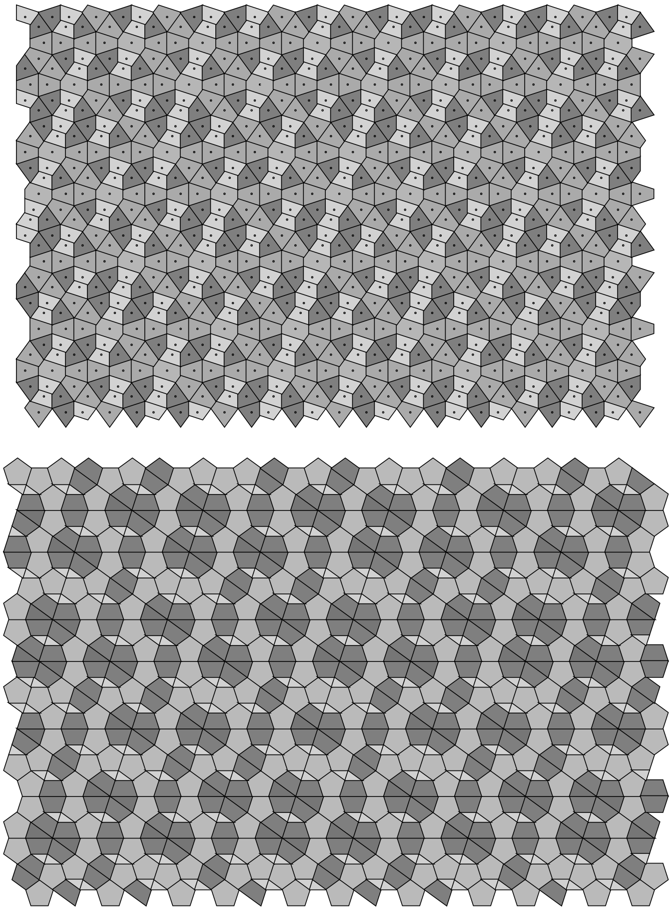


Figure 9. Example of a singular Voronoi and Delone tiling: a quasilattice whose acceptance window has the side length $d = 1$. It represents the cases VT_6 and DT_4.

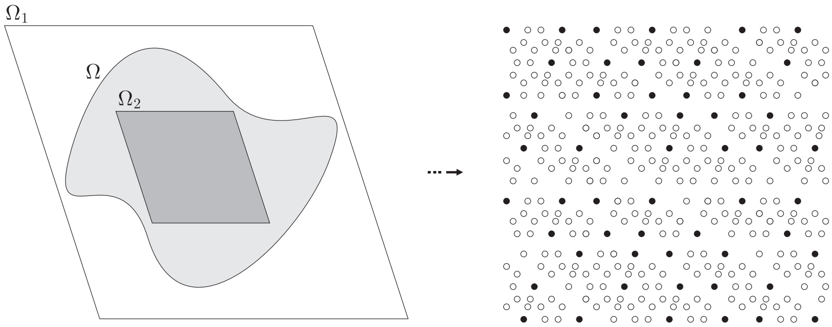


Figure 10. Example of a skeleton quasilattice. The acceptance windows Ω_1, Ω_2 and Ω are shown on the left. A fragment of the corresponding skeleton quasilattice composed of quasilattices $\Sigma(\Omega_1)$ (empty and black circles) and $\Sigma(\Omega_2)$ (black circles) is shown on the right.

Each point from any configuration from $\tilde{\mathcal{N}}$ is flagged by $\circ$ or $\bullet$ according to the following rule. The point is marked by $\bullet$ if both corresponding coordinates on axes α_1 and α_2 are also flagged by $\bullet$, otherwise the point is flagged by $\circ$.

(6) Trim each configuration from $\tilde{\mathcal{N}}$ by the disc of radius $2R_c$ centred at the origin. Finally remove duplicate configurations.

In $\tilde{\mathcal{N}}$ one finds all local configurations of required size from the two-dimensional skeleton given by Ω_1 and Ω_2. For any Voronoi tile $V(x)$ of $\Sigma(\Omega)$, one finds in $\tilde{\mathcal{N}}$ the configuration containing points $N_1(x)$ together with the centre point x. The points from $N_1(x)$, however, may be marked by $\circ$ or $\bullet$ in this configuration. Thus before we move to the final step, we construct a superset of Voronoi tiles which certainly contains all Voronoi tiles from the tiling of $\Sigma(\Omega)$. This superset is constructed using $\tilde{\mathcal{N}}$ as follows.

(7) To each configuration from $\tilde{\mathcal{N}}$ create a set of configurations that are composed by origin, points marked by $\bullet$ and all possible combinations of points marked by $\circ$; that is, if the number of points marked by $\circ$ is k we obtain 2^k possibilities. Save all these configurations to $\mathcal{N}$.
(8) From each configuration from $\mathcal{N}$ construct the Voronoi tile around the origin. Thus with each configuration from $\mathcal{N}$ we associate a Voronoi tile.
(9) Delete all points in each configuration except the origin and the neighbours N_1^1 of the associated Voronoi tile.
(10) Remove duplicate Voronoi tiles and their configurations from $\mathcal{N}$.

After these steps, each configuration in the list $\mathcal{N}$ corresponds to a different Voronoi tile. Therefore we can use 'configuration from $\mathcal{N}$' and 'Voronoi tile' as synonymous without ambiguity. In step 7 a very large list of configurations has been created. Then in step 10 it was purged.

In the list $\mathcal{N}$, there are many Voronoi tiles that do not occur in the Voronoi tiling of $\Sigma(\Omega)$. So far we have not taken into account the particular shape of the acceptance window Ω under consideration. In fact the list $\mathcal{N}$ also contains, besides the Voronoi tiles of $\Sigma(\Omega)$, Voronoi tiles appearing in every other quasicrystal $\Sigma(\tilde{\Omega})$ whose acceptance window $\tilde{\Omega}$ satisfies the inclusions $\Omega_2 \subset \tilde{\Omega} \subset \Omega_1$.

At the next step, we discard all configurations in $\mathcal{N}$ but those with the star map image in the given acceptance window Ω. Let V be a configuration from $\mathcal{N}$, and let $q_1, \ldots, q_m \in \Sigma(\Omega)$ be the positions of the neighbours that shape V. Actually $q_1, \ldots, q_m \in N_1^1$ are already stored in $\mathcal{N}$ in this configuration. Thus we want to find all positions of the images c^* of the centre point c of $V = V(c)$ such that images of neighbours lie in the acceptance window. We require that

$$c^* \in \Omega \quad \text{and} \quad c^* + q_j^\star \in \Omega \qquad \text{for} \quad j = 1, \ldots, m \tag{16}$$

which is equivalent to the condition

$$c^* \in \Omega \quad \text{and} \quad c^* \in \Omega - q_j^\star \qquad \text{for} \quad j = 1, \ldots, m.$$

Therefore c^* has to be situated in the intersection of all $\Omega - q_j^\star$. More precisely,

$$c^* \in \Omega|_V \stackrel{\text{def}}{=} \left(\bigcap_{j=1}^{m} (\Omega - q_j^\star) \right) \cap \Omega. \tag{17}$$

Geometrically $\Omega|_V$ is an intersection of a few shifted copies of Ω. If $\Omega|_V$ is empty, the tile $V(c)$ does not occur in the Voronoi tiling of $\Sigma(\Omega)$. A non-empty $\Omega|_V$ is a region in Ω which contains images of the points c such that

$$c \in \Sigma(\Omega) \quad \text{and} \quad c + q_j \in \Sigma(\Omega) \qquad \text{for} \quad j = 1, \ldots, m.$$

The condition that $\Omega|_V$ is non-empty is necessary but not sufficient for the tile V to be present in the tiling of $\Sigma(\Omega)$.

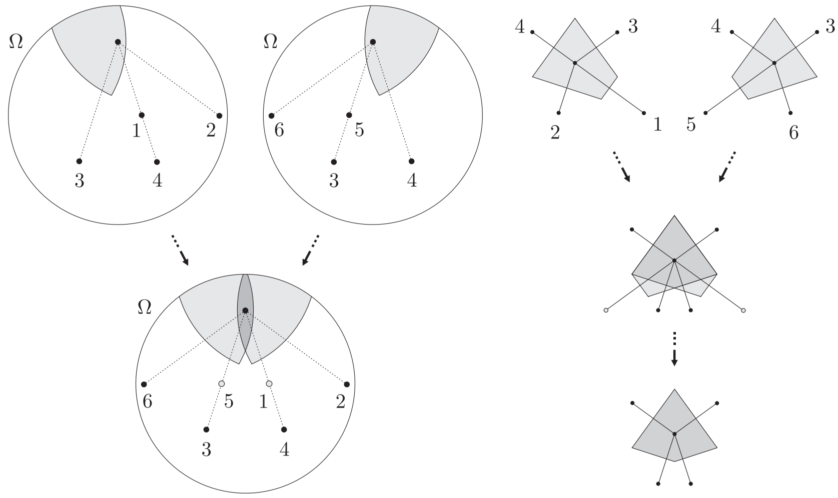


Figure 11. Example of two overlapping regions in the acceptance window (on the left), each corresponding to a different Voronoi tile (on the right). The intersection of the regions is shown together with its Voronoi tile, which has the form of the overlap of the two Voronoi tiles. This figure represents a real case for circular acceptance window Ω.

The sufficient condition will be found by considering the frequently arising situation of an overlap of two regions $\Omega|_V$ corresponding to different Voronoi tiles. Such a situation is illustrated in figure 11. Which Voronoi tile does then actually correspond to the points in the intersection? Detailed consideration of the question is rather tedious, but it leads to a simple geometrical answer [28]. Suppose that two overlapping regions contain images of points of two different tiles, say, V_1 and V_2. Then any point, say p^*, found in the intersection, is the image of $p \in \Sigma(\Omega)$ whose Voronoi tile $V(p)$ is the 'intersection' of tiles V_1 and V_2. More precisely, $V(p)$ is the intersection of the two tiles after they have been placed one on top of the other, their respective quasicrystal points coinciding, and orientations preserved. It follows that among several candidates for a Voronoi tile of a point p, $V(p)$ is the one with smallest area.

In order to obtain a picture of the division of the acceptance window Ω into regions corresponding to different tiles, it is practical to complete the algorithm by the following steps:

(11) For each Voronoi tile $V \in \mathcal{N}$, test whether $\Omega|_V$ is empty or not. If it is empty remove V from $\mathcal{N}$.
(12) Sort Voronoi tiles in $\mathcal{N}$ according to their area in decreasing order.
(13) Draw $\Omega|_V$ for each V from $\mathcal{N}$ one by one, then delete such V from $\mathcal{N}$ for which $\Omega|_V$ is completely covered by other regions.

Upon completion of the last step, $\mathcal{N}$ became the list of Voronoi tiles VT of $\Sigma(\Omega)$. Step 13 implies that the region $\Phi(V)$, which corresponds to Voronoi tile V, is given by

$$\Phi(V) = \Omega|_V \backslash \bigcup \{\Omega|_{\tilde{V}} \text{where } |\tilde{V}| \leqslant |V|\}. \tag{18}$$

It will be shown in [13, 14], for the case of a circular and decagonal window, that it is advantageous to draw the acceptance window and its division according to steps 11–13, and then visually check which subregions are covered and which are not. The regions not covered at the end represent the set of Voronoi tiles that occur in the Voronoi tiling of $\Sigma(\Omega)$, i.e. which are represented in VT.

For a description of all steps of an implementation of this algorithm see [28].

As in the previous section, we now consider the problem of finding all Delone tiles of DT from the Delone tiling of $\Sigma(\Omega)$. The solution is analogous to that presented for the case of quasilattices. The main difference is in step 9, where we do not delete degenerate neighbours from N_1^0. Thus after step 10, there could be multiple different configurations in the list $\mathcal{N}$ which correspond to one Voronoi tile. The rest of the steps remain unchanged except for step 12. In the present problem, there could be multiple Voronoi tiles with minimal area. Since we want to have Voronoi tiles with a large number of neighbours on the top (when the acceptance window and its division are drawn), an additional ordering rule is added to step 12: if two Voronoi tiles have the same area, the Voronoi tile with the larger number of neighbours is the second one to be drawn.

After all the steps are performed, we obtain the same division of the acceptance window Ω as in the previous case, except that some of the subregion $\Phi(V)$ may be further divided into smaller parts. These parts correspond to the same Voronoi tile but different configurations of degenerate neighbours. A Voronoi tile V in the final list $\mathcal{N}$ may subsequently occur here in a few variations depending on the degenerate neighbours. But that was the main purpose of this modification. Now we know all Voronoi tiles together with all possible configurations of degenerate neighbours. Thus the rest of our task is to construct, around each vertex of every Voronoi tile, a Delone tile according to the algorithm of the previous section. This process gives us the complete list DT of Delone tiles from the Delone tiling of $\Sigma(\Omega)$.

6. Concluding remarks

(1) It appears to be a general fact, independent of the shape of the window considered here, that there are fewer DT tilings than VT ones. Similarly, for a circular window (any radius) there are 22 sets of VT and 8 sets of DT [13]; for the regular decagon as the quasicrystal acceptance window, we have determined that there are 11 VT tiling sets and 4 DT ones, see [14]. An exhaustive description of the theory and computing procedures involved in the method presented here, can be found in [28].

(2) Calling the point sets of this paper two-dimensional quasicrystals is an idealization based on two assumptions (besides the obvious dependence on the form of the acceptance window). The first one appears to be the more plausible of the two: it is the validity of the cut-and-project method in general. Our second assumption is on the points we choose to project, namely the points of the root lattice of type A_4. Our justification is the mathematical simplicity of the classification problem that we want to solve. The price we pay for it is that we do not find some of the tilings familiar from the literature, in particular any of the Penrose tilings. (An explicit prescription for the construction of the rhombic Penrose tiling by means of the cut-and-project method is found in [20].) Somewhat more complicated choices of points for the projection, still bound to the A_4-symmetry, would be so-called holes of the root lattice or vertices of the Voronoi domains of the root or weight lattices. In physics, the holes of a lattice are the points most distant from the lattice points. In Lie theory, the lattice is the lattice of roots (of type A_4 in our case), while the holes together with the root lattice form the lattice of A_4-weights. The four types of A_4-holes are precisely the four non-zero congruence classes of A_4-weights (see for example [10]).

(3) Application of the method to cut and project point sets with other quadratic irrationalities than τ, is fairly straightforward for the series of quadratic Pisot numbers

given by the solution of algebraic equations $x^2 = mx \pm 1$ in [12]. The present case, equation $x^2 = x + 1$, is the lowest member of that series.

(4) With the classification of tiles completed, it is possible to answer a number of related questions, which occasionally may be of independent interest:

- What are the lengths of tile edges in a given quasicrystal? What are their relative densities as functions of the size of the window?
- What are the relative densities of various tiles? Which tile has the maximal density? What is the value of the size of the window, where the highest density is achieved?
- Putting several adjacent tiles into a cluster, can just a few clusters be formed so that they provide a tiling of the entire plane $\mathbb{R}^2$? Examples of such a possibility are known from Penrose tilings.

Acknowledgments

We are grateful to E Pelantová for useful suggestions. ZM and JZ are grateful to Centre de recherches mathématiques, Université de Montréal, where a part of the work was done, for the hospitality. We acknowledge the financial support of the Grant Agency of the Czech Republic GA ČR 201/01/0130, the National Science and Engineering Research Council of Canada, and the FCAR of Québec.

References

[1] Baake M, Ben-Abraham S I, Klitzing R, Kramer P and Schlottmann M 1994 Classification of local configurations in quasicrystals *Acta Cryst.* A **50** 553–66

[2] Baake M and Joseph D 1990 Ideal and defective vertex configurations in the planar octagonal quasilattice *Phys. Rev.* B **42** 8091–102

[3] Baake M, Kramer P, Schlottmann M and Zeidler D 1990 Planar patterns with fivefold symmetry as sections of periodic structures in 4-space *Int. J. Mod. Phys.* B **4** 2217–68

[4] Ben-Abraham S I 1993 Defective vertex configurations in quasicrystalline structures *Int. J. Mod. Phys.* B **7** 1415–25

[5] Chen L, Moody R V and Patera J 1998 Noncrystallographic root systems *Quasicrystals and Discrete Geometry (Fields Institute Monograph Series, Vol 10)* ed J Patera (Providence, RI: American Mathematical Society) pp 143–88

[6] Shephard G 1986 *Tilings and Patterns* (New York: Freeman)

[7] Guimond L S, Masáková Z and Pelantová E 2003 Combinatorial properties of infinite words associated with cut-and-project sequences *J. Théor. Nombres Bordeaux* at press

[8] Katz A and Duneau M 1986 Quasiperiodic patterns and icosahedral symmetry *J. Physique* **47** 181–96

[9] Kramer P 2001 Delone clusters and coverings for icosahedral quasicrystals *J. Phys. A: Math. Gen.* **34** 1885–902

[10] Lemire F W and Patera J 1980 Congruence number, a generalisation of $SU(3)$ triality *J. Math. Phys.* **21** 2026–7

[11] Masáková Z, Patera J and Pelantová E 1998 Minimal distances in quasicrystals *J. Phys. A: Math. Gen.* **31** 1539–52

[12] Masáková Z, Patera J and Pelantová E 2000 Lattice-like properties of quasicrystal models with quadratic irrationalities *Proc. Quantum Theory and Symmetries (Goslar, 1999)* ed H D Doebner, V K Dobrev, J D Hennig and W Luecke (Singapore: World Scientific) pp 499–509

[13] Masáková Z, Patera J and Zich J 2002 Classification of Voronoi and Delone tiles in quasicrystals: II. Circular acceptance window of arbitrary size **36** 1895–911

[14] Masáková Z, Patera J and Zich J 2003 Classification of Voronoi and Delone tiles in quasicrystals: III. Decagonal acceptance window of arbitrary size *Preprint* CRM

[15] Moody R V 1996 Meyer sets and their duals *Mathematics of Long Range Aperiodic Order, Proc. NATO ASI (Waterloo, 1996)* ed R V Moody (Dordrecht: Kluwer) pp 403–41

[16] Moody R V and Patera J 1993 Quasicrystals and icosians *J. Phys. A: Math. Gen.* **26** 2829–53

[17] Moody R V and Patera J 1998 Densities, minimal distances, and coverings of quasicrystals *Commun. Math. Phys.* **195** 613–26

[18] Moody R V and Patera J 1992 Voronoi and Delone cells of root lattices: classification of their faces and facets by Coxeter–Dynkin diagrams *J. Phys. A: Math. Gen.* **33** 1618–26
[19] Moody R V and Patera J 1995 Voronoi domains and dual cells in the generalized kaleidoscope with applications to root and weight lattices *Can. J. Math.* **47** 573–605
[20] Moody R V and Patera J 1994 Colorings of quasicrystals *Can. J. Phys.* **72** 442–52
[21] Okabe A, Boots B and Sugihara K 1992 *Spatial Tessellations, Concepts and Applications of Voronoi Diagrams* (Chichester: Wiley)
[22] Sadoc J F and Mosseri R 1993 The E_8 lattice and quasicrystals: geometry, number theory and quasicrystals *J. Phys. A: Math. Gen.* **26** 1789–809
[23] Schulte E 1993 Tilings *Handbook of Convex Geometry* ed P M Gruber and J M Willis (Amsterdam: Elsevier)
[24] Senechal M 1995 *Quasicrystals and Geometry* (Cambridge: Cambridge University Press)
[25] Shechtman D, Gratias I and Cahn J W 1984 Metallic phase with long range orientational order and no translational symmetry *Phys. Rev. Lett.* **53** 1951–3
[26] Sloane N J A and Conway J 1988 *Sphere Packings, Lattices and Groups* (Berlin: Springer)
[27] Sloane N J A and Conway J 1991 The cell structures of certain lattices *Miscelanea Mathematica* ed P Hilton, F Hirzebruch and R Remmert (Berlin: Springer)
[28] Zich J 2002 Voronoi and Delone tilings of quasicrystals *MSc Thesis* Czech Technical University, Prague